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The wave equation

$$\nabla^2 u = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \quad (1) \qquad \nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

a Partial Differential Equation

If we assume that the field variation is sinusoidal

$$\underset{\text{The wave}}{u(\vec{R}, t)} = \underbrace{u_0(\vec{R})}_{\text{complex amplitude of the wave}} e^{-j\omega t} \quad (2)$$

If we insert (2) into (1)

$$\boxed{\nabla^2 u_0 + k^2 u_0 = 0}$$

The Helmholtz's Equation (3)

, where

$$k = \frac{\omega}{c} = \frac{2\pi}{\lambda}$$ optical wave number (1/m)

For optical wave propagation we can further reduce the Helmholtz's eqn:

assumptions:

- beam originates at $z=0$ (in the $z=0$ plane)
- beam propagates in z axis
- symmetric along the z axis (rotationally symmetric)

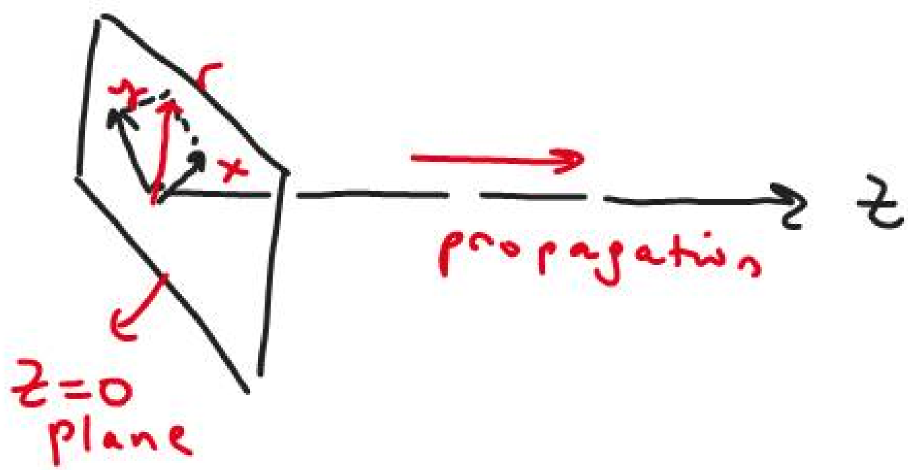
then it can be expressed as a function of r and z

$r = \sqrt{x^2+y^2}$ and z

Thus we reduce equation (3) in cylindrical coordinates as

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_0}{\partial r}\right)+\frac{\partial^2 u_0}{\partial z^2}+k^2 u_0=0 \qquad (4)$$

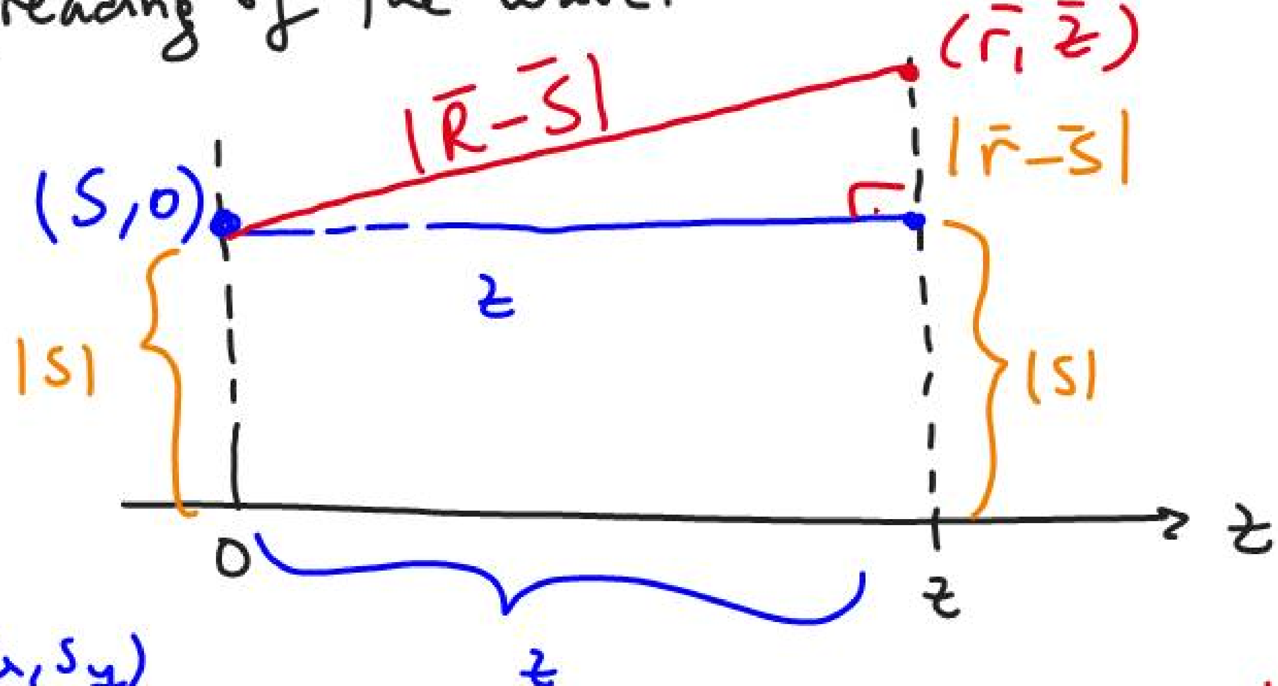
Lets use $u_0(r,z)=V(r,z)\cdot e^{ikz}$

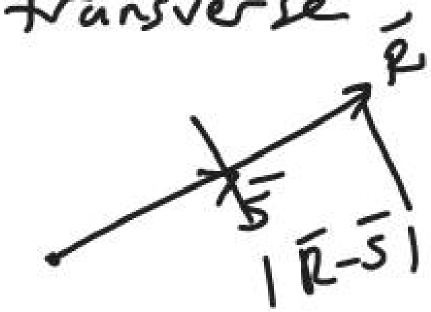


Eqn (4) becomes $\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right)+\frac{\partial^2 V}{\partial z^2}+2ik\frac{\partial V}{\partial z}=0 \qquad (5)$

To further simplify (5) we use the Paraxial Approximation

It says the propagation distance for an optical wave along z-axis is much greater than the transverse spreading of the wave.





$S = (S_x, S_y)$

$|\bar{r}-\bar{s}| \ll z$

Paraxial approx.

$$|\bar{R}-\bar{S}| = (z^2 + |r-s|^2)^{1/2}$$

$$|\bar{R}-\bar{s}| = z\left(1+\frac{|\bar{r}-\bar{s}|^2}{z^2}\right)^{1/2}$$

$$|\bar{R}-\bar{s}| = z\left(1+\frac{1}{2}\,\frac{|\bar{r}-\bar{s}|^2}{z^2}+\cdots\right)$$ binomial expansion

$$\boxed{|\bar{R}-\bar{s}| \cong z + \frac{|\bar{r}-\bar{s}|^2}{2z} \qquad |\bar{r}-\bar{s}| \ll z}$$

The paraxial approximation (6)

This leads to

$$\left|\frac{\partial^2 V}{\partial z^2}\right| \ll \left|2k\frac{\partial V}{\partial z}\right|$$ { diffraction effects: it changes slowly in the z direction

$$\left|\frac{\partial^2 V}{\partial z^2}\right| \ll \left|\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right)\right|$$ { due to the finite size of the beam

These set of equations permit us to use

$$\frac{\partial^2 V}{\partial z^2} = 0 \text{ in Eqn (5).}$$

$$\boxed{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + 2ik\frac{\partial V}{\partial z} = 0} \quad (8)$$

Parabolic Wave Equation or

Paraxial Wave Equation

2 SOLUTIONS

The Direct Method

Huygens-Fresnel Integral Method

Question :

$$|r| = \sqrt{x^2+y^2}$$

If at $z=0$ (source plane)

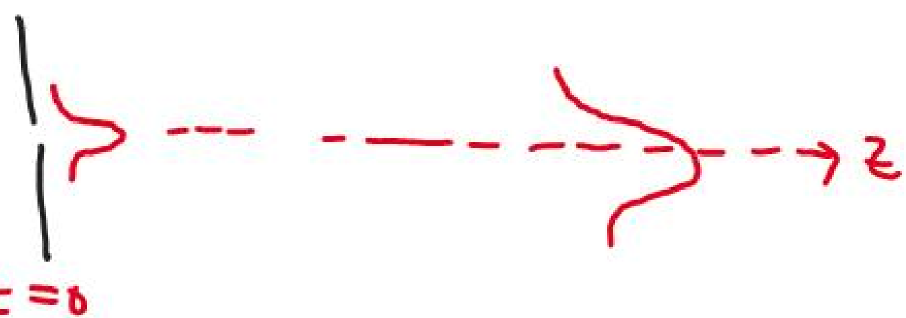
$$V(r,0) = a_0 \exp\left(-\frac{r^2}{w_0^2} - \frac{ikr^2}{2F_0}\right)$$

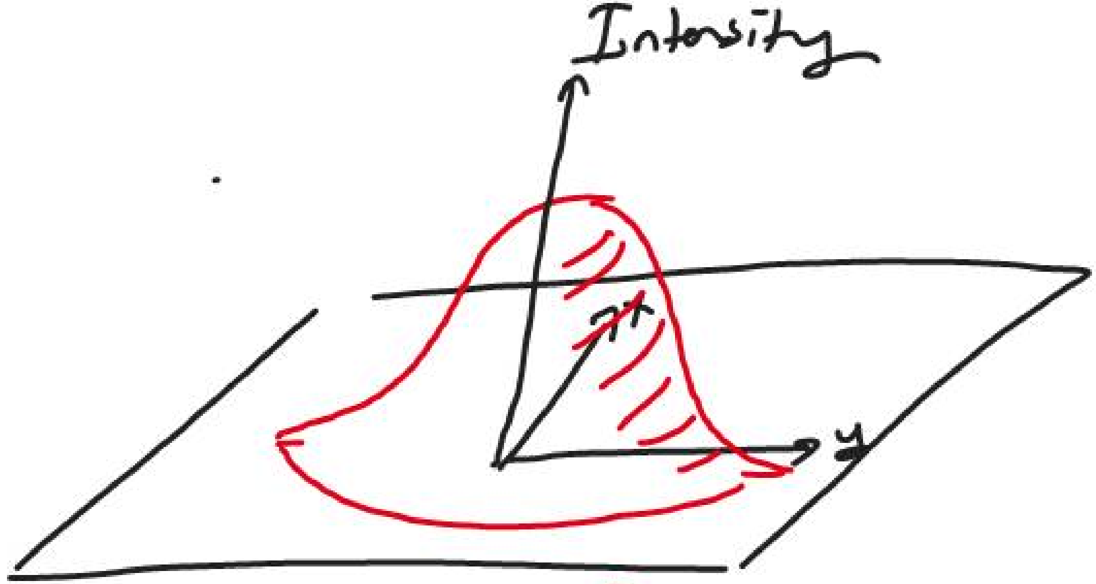
(the Gaussian beam)

W_0: spot size
F_0: focal length

Then calculate $V(r,z)$

a) Using direct method
b) using the Huygens-Fresnel Integral.

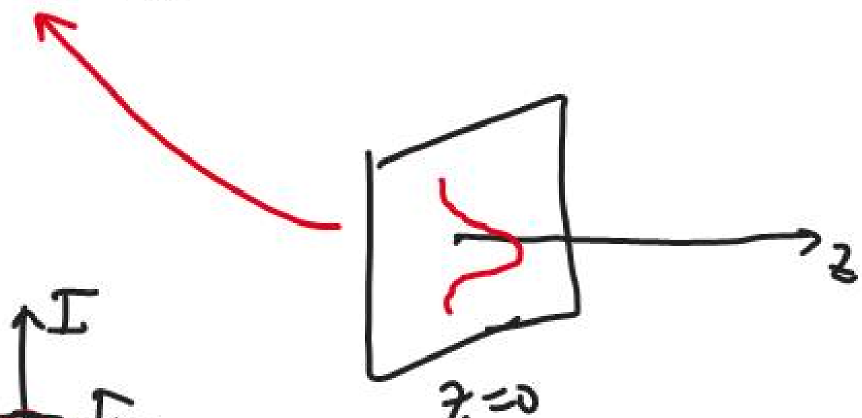




Intensity $= U^* \cdot U$

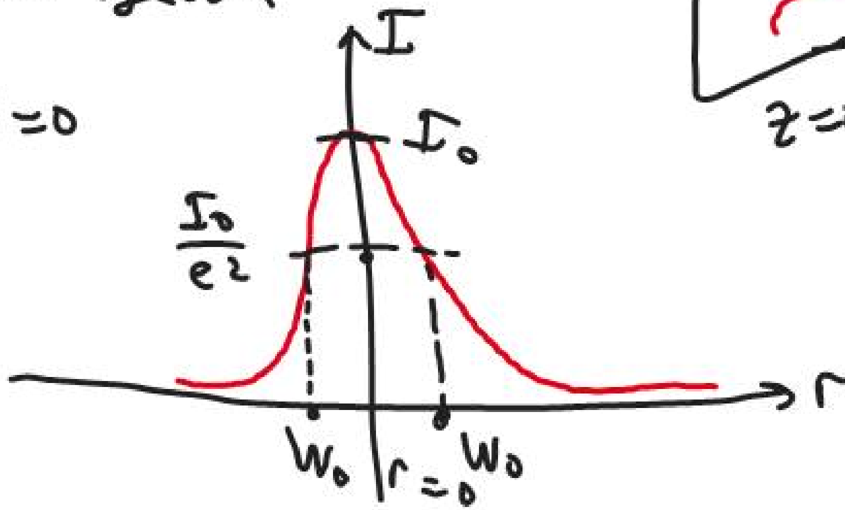
(watt/m²)

at $z=0$



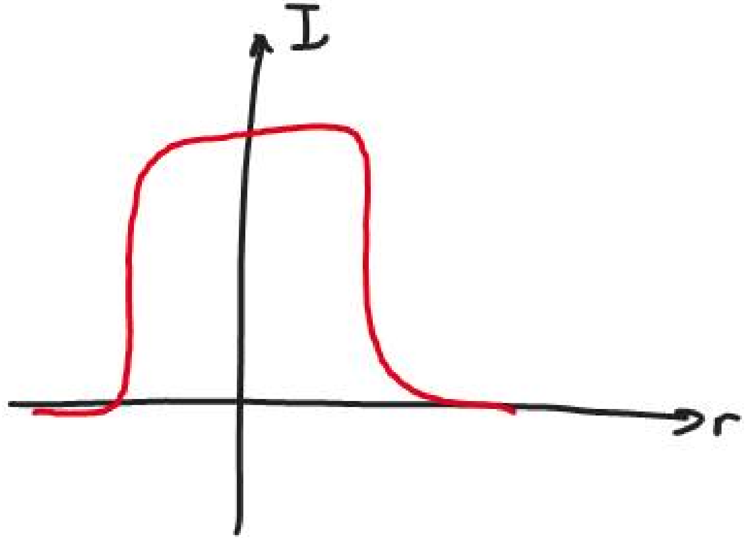
Gaussian Beam

$z=0$

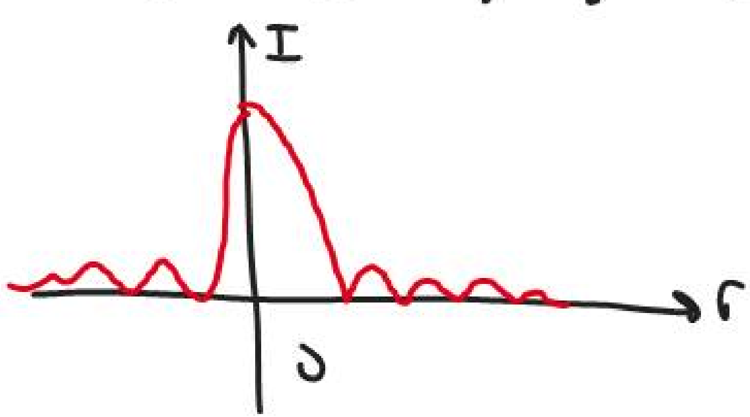


$$I = I_0 e^{-\frac{2r^2}{w_0^2}}$$

for cutting purposes: Flat-top beam



for communication: Bessel Beams in the Free Space Optics (FSO) systems



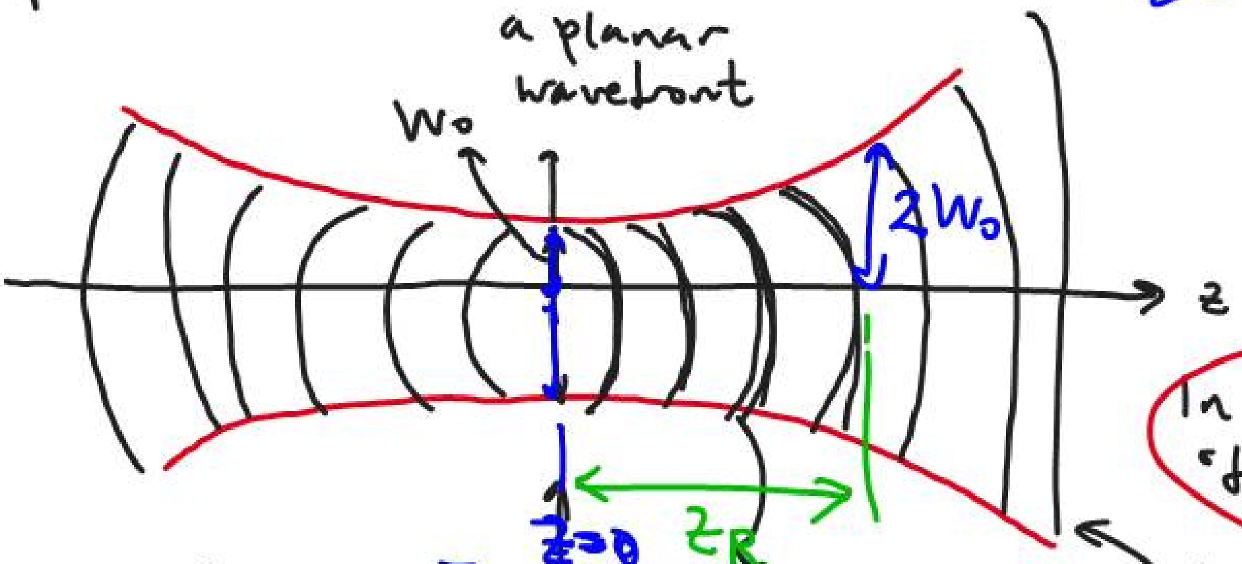
LASER BEAM PROPERTIES

The beam from a LASER expands as though emanating from a point source

at Rayleigh range

W_0: spot radius at $z=0$

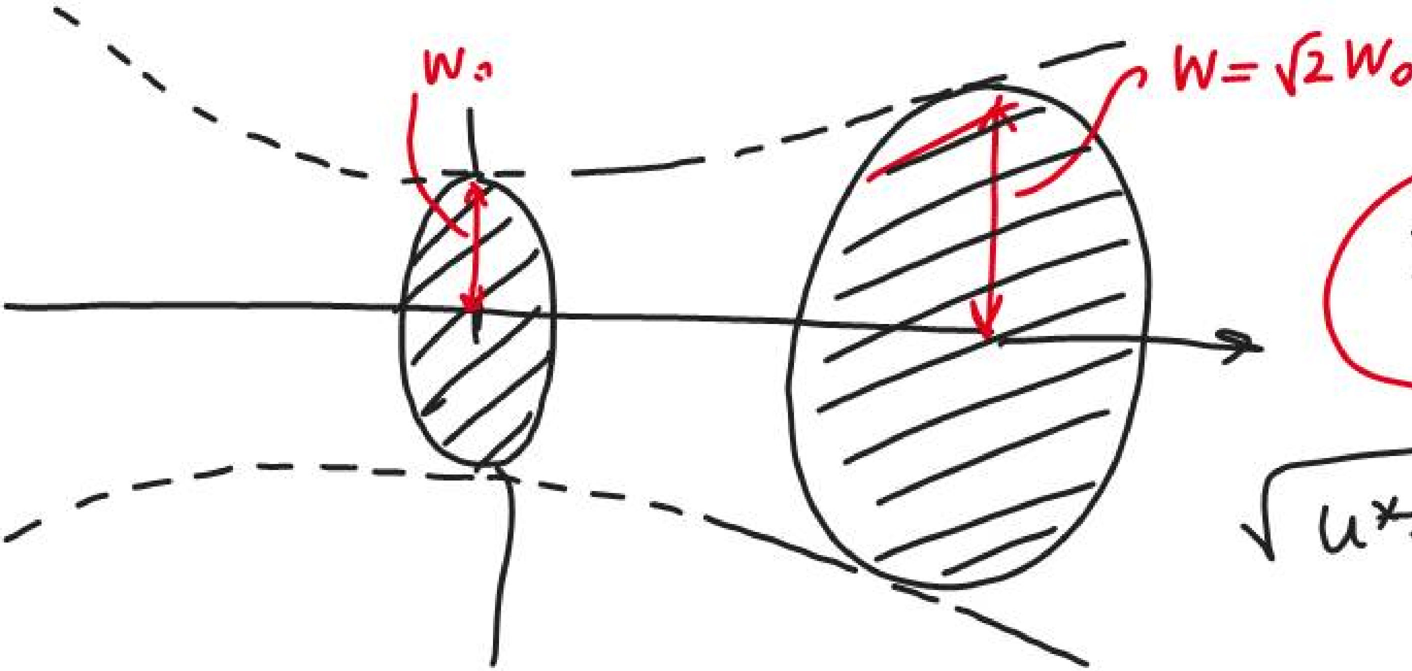
$z = z_R \rightarrow W(z_R) = 2W_0$



In terms of intensity $\propto u^{*} \cdot u$

$$W(z) = W_0 \left[1+\left(\frac{z}{z_R}\right)^2\right]$$

$$\boxed{z_R = \text{Rayleigh Range} = \frac{\pi W_0^2}{\lambda}}$$



For the amplitude of the beam

$\sqrt{u^* \cdot u}$

Area of the beam $= \pi W_0^2$ at the waist $(z=0)$

$2\pi W_0^2$

$$W(z) = W_0\left[1+\left(\frac{z}{z_R}\right)^2\right]^{1/2}$$

at $z = z_R$

$$W(z) = \sqrt{2}\,W_0$$

Beam Amplitude versus Intensity

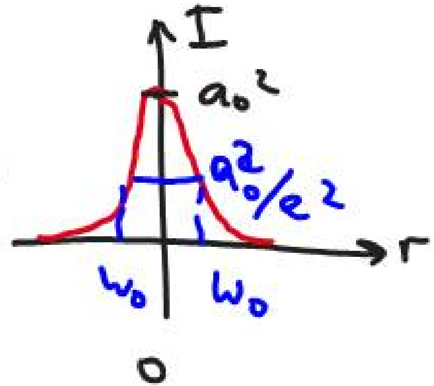
$$V(r,0) = a_0 \exp\left(-\frac{r^2}{w_0^2} + \frac{ikr^2}{2F_0}\right)$$

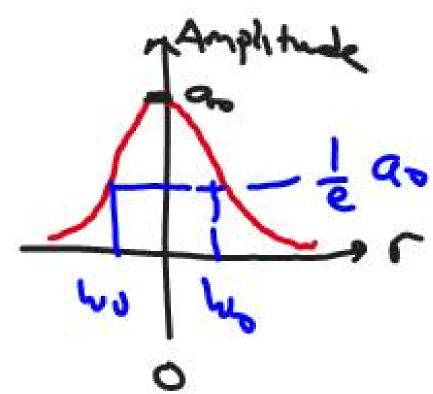
$$I = V^* V = a_0^2 \exp\left(-\frac{r^2}{w_0^2} - \cancel{\frac{ikr^2}{2F_0}} - \frac{r^2}{w_0^2} + \cancel{\frac{ikr^2}{2F_0}}\right)$$

$$I = a_0^2 \exp\left(-\frac{2r^2}{w_0^2}\right) \qquad (W/m^2)$$

$$\text{Field Amplitude} = \sqrt{V^* V} = a_0 \exp\left(-\frac{r^2}{w_0^2}\right) \quad (V/m)$$

$$\left\{ \begin{array}{l} W = V \times I = V \cdot \frac{V}{R} = \frac{V^2}{R} \\ \text{Watt} \propto \text{Volt}^2 \\ \qquad \propto \text{Ampere}^2 \end{array} \right\}$$





$z > z_R$ Far field region of the beam

$z < z_R$ Near field region of the beam

$$\boxed{z_R = \frac{\pi w_0^2}{\lambda}} = \frac{\text{beam area at the waist}}{\text{wavelength}}$$

Total power in the beam:

$$I(r) = I_0 \exp\left(-\frac{2r^2}{w_0^2}\right)$$

Half of the beam area

$$P = \int I(r)\, 2r\pi\, dr = \frac{\pi w_0^2 \, I_0}{2}$$

→ Intensity at $r=0$