EINSTEIN COEFFICIENTS

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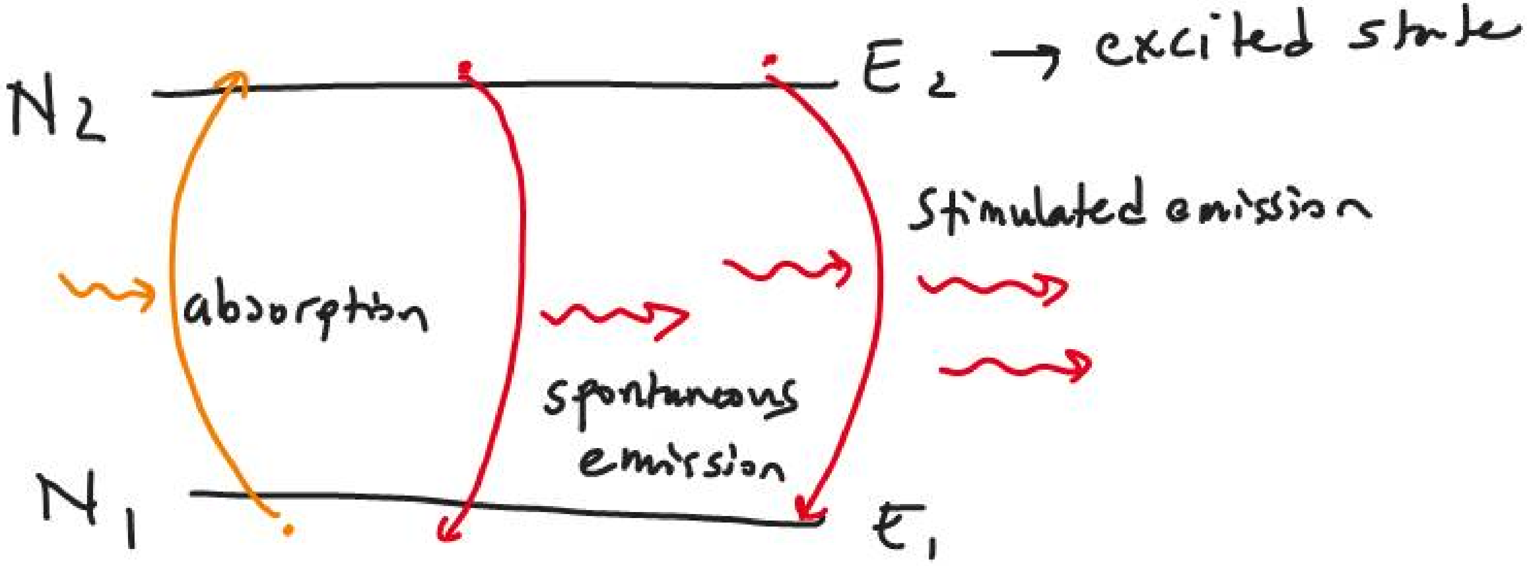
In a 2-level system:



The rate of spontaneous Emission = $N_2 A_{21}$

N_2: # of atoms per unit volume in the upper state E_2 $\left(\frac{1}{m^3}\right)$

A_{21}: the probability per unit time that a given atom in E_2 will spontaneously undergo a transition to E_1 (1/sec)

$N_2 A_{21}$ = rate per unit volume $\left(\frac{1}{m^3\,sec}\right)$

For spontaneous emission we do not need a radiation field.

However, absorption and stimulated emission need a radiation field.

ABSORPTION : $N_1 \cdot \rho_\nu \cdot B_{12} \Rightarrow$ Rate of absorption per unit volume $\left(\frac{1}{m^3\,sec}\right)$

ρ_ν = energy density $\Rightarrow \left(\frac{J}{m^3\,Hz}\right) \rightarrow \left(\frac{J\,sec}{m^3}\right)$ $Hz = \frac{1}{sec}$

(energy per volume per Hz)

B_{12}: probability per unit time per unit spectral energy density that a given atom will be excited from E_1 to E_2.

$$B_{12} \rightarrow \left(\frac{m^3\,Hz}{J\ sec}\right) \qquad \frac{1}{\underbrace{\boxed{\frac{J}{m^3\,Hz}}}_{\text{energy density}} \cdot \underbrace{\boxed{sec}}_{\text{time}}}$$

$$N_1 \cdot \rho_\nu \cdot B_{12} = \left(\frac{1}{\cancel{m^3}}\right)\left(\frac{\cancel{J}}{m^3\,\cancel{Hz}}\right)\left(\frac{\cancel{m^3}\,\cancel{Hz}}{\cancel{J}\ sec}\right) \rightarrow \left(\frac{1}{m^3\,sec}\right)$$

STIMULATED EMISSION Rate : $N_2 \cdot \rho_\nu \cdot B_{21} \quad \left(\frac{1}{m^3\,sec}\right)$

<u>At equilibrium</u>, assuming only optical transitions occur

ABSORPTION RATE = SPONTANEOUS EMISSION RATE + STIMULATED EMISSION RATE

$$\frac{N_1 \cdot \rho_\nu \cdot B_{12}}{N_1} = \frac{N_2 \cdot A_{21}}{N_1} + \frac{N_2 \cdot \rho_\nu \cdot B_{21}}{N_1}$$

$$\rho_\nu B_{12} = \frac{N_2}{N_1} \cdot A_{21} + \frac{N_2}{N_1} \rho_\nu B_{21}$$

$$\frac{N_2}{N_1} = \exp\left(-\frac{E_2 - E_1}{kT}\right)$$ Boltzmann's Statistics

$$E_2 - E_1 = h\nu_{21}$$

$$\frac{N_2}{N_1} = \exp\left(-\frac{h\nu_{21}}{kT}\right)$$

$$\rho_\nu B_{12} = A_{21} e^{-\frac{h\nu_{21}}{kT}} + \rho_\nu B_{21} \cdot e^{-\frac{h\nu_{21}}{kT}}$$

$$\rho_\nu \left(B_{12} - B_{21} e^{-h\nu_{21}/kT}\right) = A_{21} e^{-\frac{h\nu_{21}}{kT}}$$

$$\rho_\nu = \frac{A_{21}\, e^{-h\nu/kT}}{B_{12}\left(1 - e^{-h\nu/kT}\right)}$$

If we take $B_{12} = B_{21}$

$$\rho_\nu = \frac{A_{21}}{B_{12}\left(e^{h\nu/kT} - 1\right)}$$

Black Body radiation's (Planck) spectral density is:

$$\rho_\nu = \frac{8\pi h\nu^3}{c^3} \frac{1}{\left(e^{h\nu/kT} - 1\right)}$$

$$\Rightarrow \boxed{\frac{A_{21}}{B_{12}} = \frac{8\pi h\nu^3}{c^3}}$$

Einstein's Relation

$\nu = 2 \times 10^{14}\ Hz$

$h = 6.62 \times 10^{-34}\ J/Hz$

$c = 3 \times 10^{8}\ m/s$

$$\frac{A_{21}}{B_{12}} = \frac{8\pi h \nu^3}{c^3}$$

(B_{21})

$$= \frac{8\pi \times 6.62\times10^{-34} \left(J/\cancel{Hz} \right) \times \left(2\times10^{14}\,\cancel{Hz}\right)^3}{(3\times10^{8})^3 \quad m^3/s^3}$$

$$= \frac{8\pi \times 6.62\times10^{-34} \times 8 \times \overset{10^{18}}{\cancel{10^{42}}}}{27 \times \cancel{10^{24}}} \frac{\left(\frac{J}{Hz} \cdot Hz^3\right)}{\left(\frac{m^3}{s^3}\right)}$$

$$\cong 13 \times 10^{-16} \left(\frac{J\,\cancel{Hz^2}}{m^3 \cdot \cancel{Hz^3}} \right)$$

$$\frac{A_{21}}{B_{21}} \cong 13\times10^{-16} \left(\frac{J}{m^3\,Hz} \right)$$

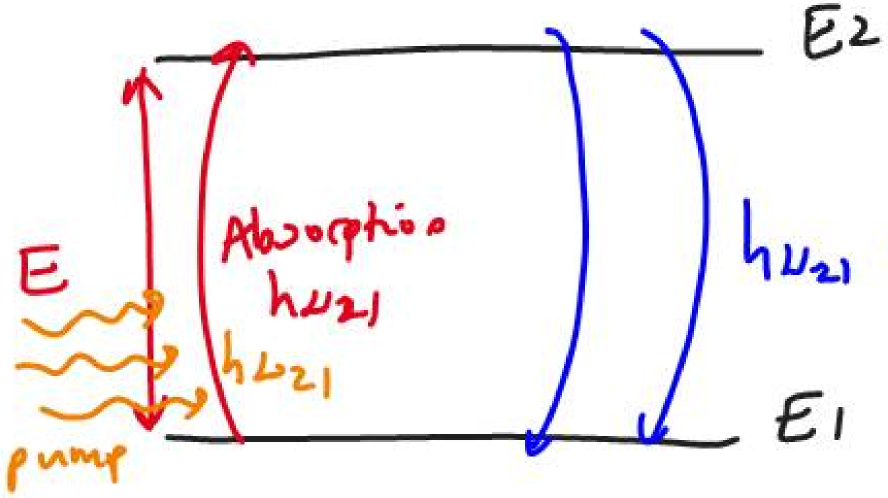
(B_{12})

$\hbar = \frac{h}{2\pi}$

$E = h\nu$

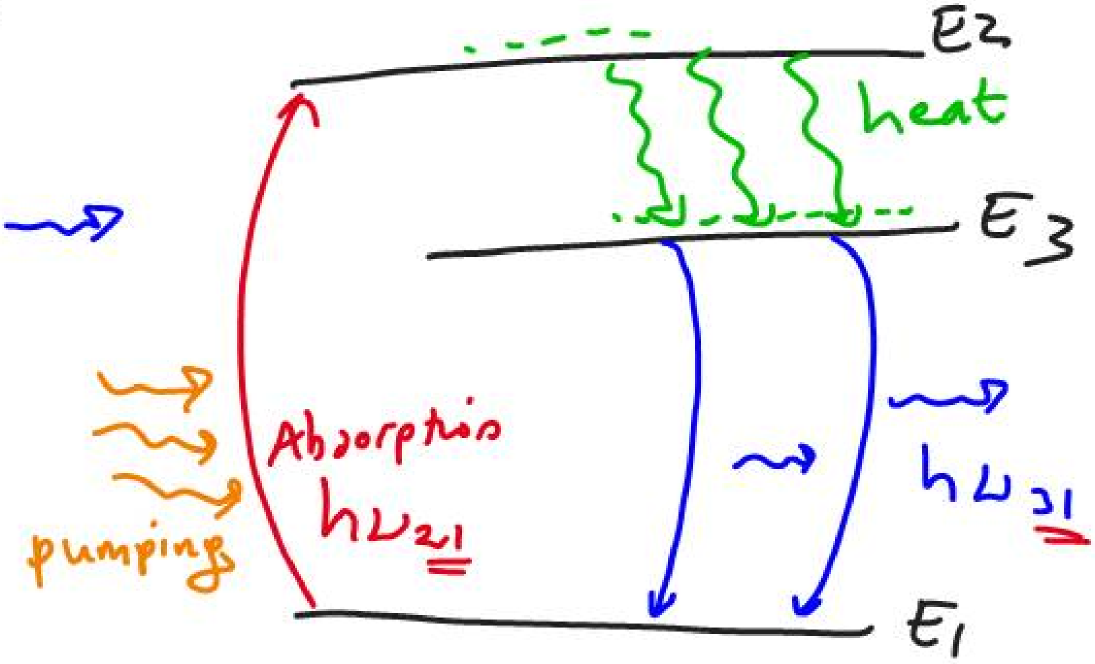
$E = \hbar\omega$

Reminder:



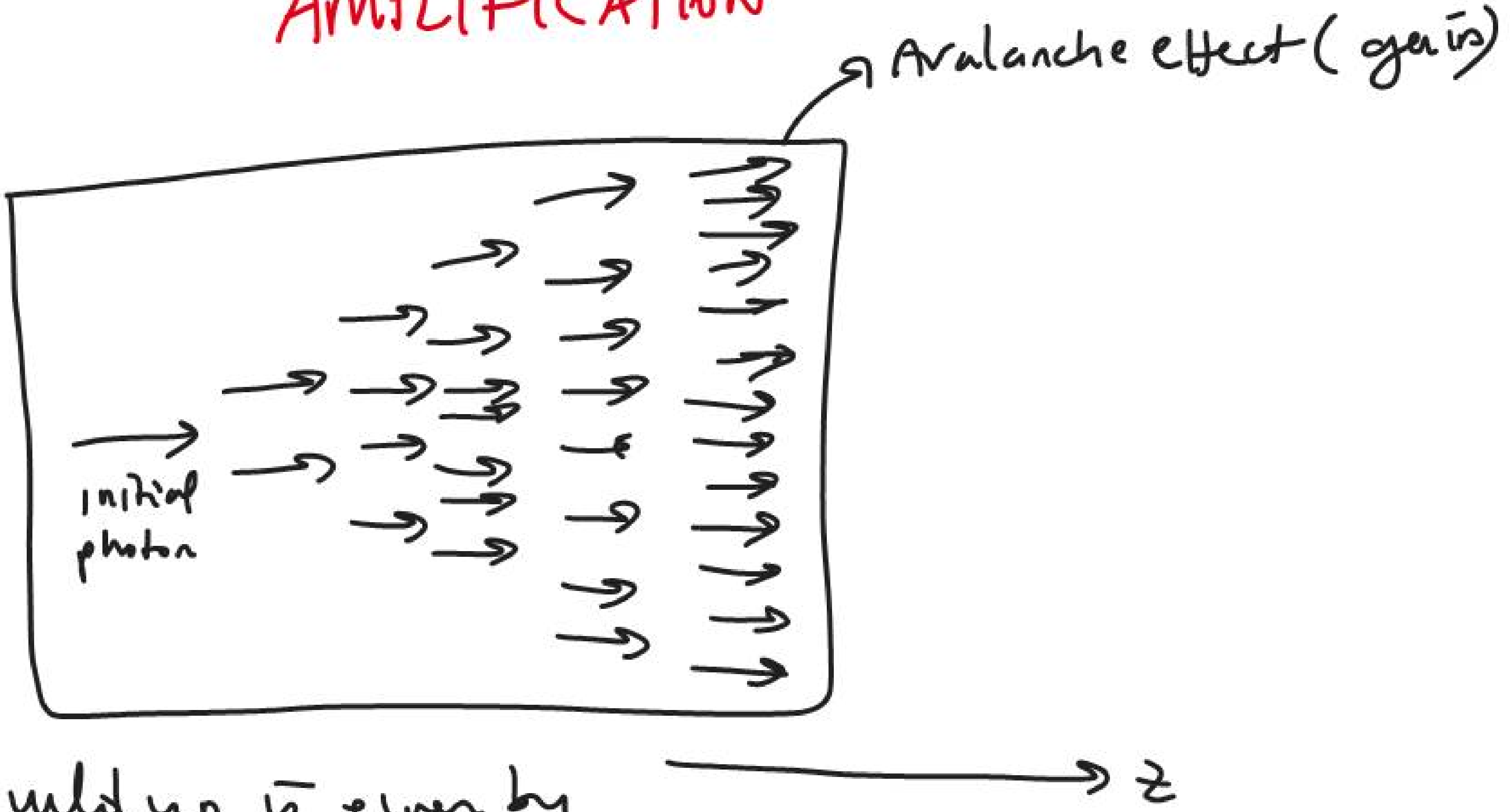
$E = E_2 - E_1 = h\nu_{21}$

3-level system



$h\nu_{21} > h\nu_{31}$

AMPLIFICATION



Build up is given by

$$I = I_0 e^{\beta z}$$

$\left(\frac{Watt}{m^2}\right)$ $\beta \left(\frac{1}{m}\right)$

at the same time some of the photons are absorbed, and due to the defects of mirrors and the imperfections in the resonator some of the photons are lost.

$$\boxed{I = I_0 e^{-\alpha z}} \quad \text{due to loss}$$

since gain and loss are simultaneously occur:

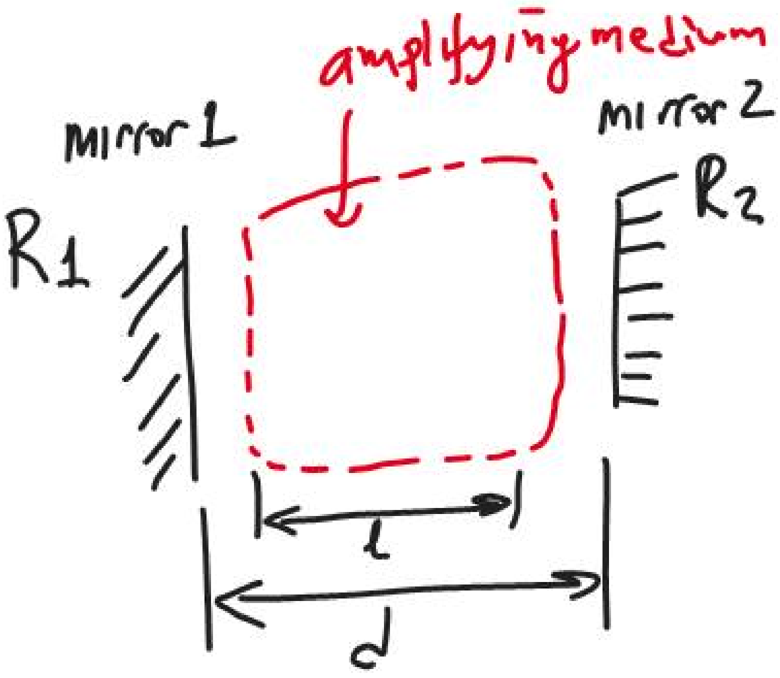
$$I = I_0 \left(e^{\beta z} \cdot e^{-\alpha z} \right)$$

$$\boxed{I = I_0 e^{(\beta - \alpha) z}}$$

The round-trip gain:

$$\boxed{G = \frac{I}{I_0} = \underbrace{R_1 \cdot R_2}_{\text{mirrors}} \; e^{(\beta-\alpha)\cdot 2l}}$$

not d
but l



2l: round trip distance in active medium

R_1, R_2 are reflectivities

$$\boxed{G = e^{(\beta-\alpha')\cdot 2l}}$$

$$e^{-\alpha'\cdot 2l} = R_1 R_2 e^{-2\alpha l}$$

Calculating the gain (β) → Small signal gain coefficient

Assuming that the spontaneous emission is negligible to the stimulated emission (neglect $A_{21} \cdot N_2$)

$$\Big(\underbrace{N_2\, \rho_\nu \cdot B_{21}}_{\text{Stimulated emission}} - \underbrace{N_1\, \rho_\nu\, B_{12}}_{\text{absorption}}\Big) \underbrace{A \cdot \Delta z}_{\text{Volume}} \cong (N_2 - N_1)\, \rho_\nu\, B_{21}\, A\, \Delta z$$

$$B_{12} \cong B_{21}$$

$$\frac{\text{Power}}{\text{Area}} = \text{Intensity}$$

$$P = \text{Power} = \text{Intensity} \cdot \text{Area} = \Delta I_\nu \cdot A$$

$$(N_2-N_1)\,\rho_\nu B_{21}\cdot\cancel{A}\cdot\Delta z=\frac{\Delta I_\nu\cdot\cancel{A}}{h\nu}$$

$$\frac{\#\text{ of atoms}}{\cancel{m^3}}\cdot\frac{\cancel{J}}{\cancel{m^3}\cdot\cancel{Hz}}\cdot\left(\frac{1}{\frac{\cancel{J}}{\cancel{m^3}\,\cancel{Hz}}\cdot sec}\right)\cdot\cancel{m^2\cdot m}$$

$$\frac{\text{Power}/\cancel{m^2}\times\cancel{m^2}}{\text{energy}}$$

$$\frac{\#\text{ Photons}\times\overset{\text{photon}}{\cancel{\text{energy}}}/\text{sec}}{\text{photon }\cancel{\text{energy}}}$$

$$\frac{1}{\text{sec}}\,\#\text{ of photons}$$

$$\frac{\#\text{ of atoms}}{sec}\Rightarrow\frac{\#\text{ of photons}}{sec}$$

$$\frac{\Delta I_\nu}{\Delta z}=(N_2-N_1)\,\rho_\nu B_{21}\,h\nu$$

$$\Delta z\to 0$$

$$\frac{dI_\nu}{dz}=(N_2-N_1)\,\rho_\nu B_{21}\,h\nu$$

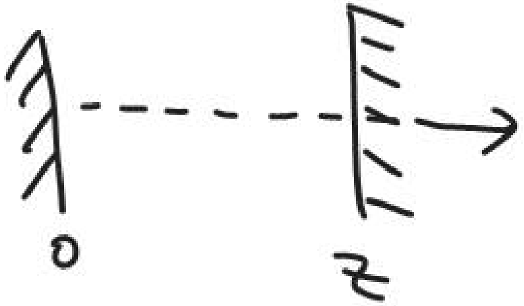
For a unidirectional (linear) beam

$$I_\nu = \frac{\rho_\nu \cdot c}{\tau} \qquad \frac{J}{{}^{2}m^3\,\cancel{Hz}\,\cancel{sec}} \cdot \frac{m}{sec} = \frac{J}{m^2\,sec} = \frac{watt}{m^2} \;\checkmark$$

$$\rho_\nu = \frac{I_\nu \cdot \tau}{c}$$

$$\frac{dI_\nu}{dz} = \frac{(N_2 - N_1)\, I_\nu\, B_{21}\, h\nu}{c}\,\tau$$

$$\int_{I_0}^{I_\nu} \frac{dI_\nu}{I_\nu} = \int_0^z \frac{(N_2 - N_1)\, B_{21} h\nu \tau}{c}\, dz$$



$$\underset{\cancel{e}}{\cancel{\ln}}\ \frac{I_\nu}{I_0} = e^{\frac{(N_2 - N_1)\, B_{21} h\nu \tau}{c} z}$$

$$I = I_0\, e^{\frac{[(N_2-N_1)B_{21}\cdot h\nu\cdot\tau}{c}\cdot z}$$

$$I = I_0\, e^{\beta z}$$

$\beta(\frac{1}{\cancel{m}})\ z(\cancel{m}) \Rightarrow$ unitless ✓

$$\boxed{\beta = \frac{(N_2-N_1)B_{21}\cdot h\nu\,\tau}{c}} \qquad \left(\frac{1}{m}\right)$$

Recall that

$$A_{21} = \frac{8\pi h\nu^3}{c^3}B_{21} \Rightarrow \beta = \frac{(N_2-N_1)\cdot c^2\cdot\tau}{8\pi\nu^2}\cdot A_{21}$$

$$A_{21} = \frac{1}{\tau}$$

$$\beta = \frac{(N_2-N_1)c^2}{8\pi\nu^2}$$

$$\boxed{\beta = \frac{(N_2 - N_1)\, c^2}{8\pi \nu^2}}$$

Small signal gain coefficient

$$\frac{\left(\frac{1}{m^3}\right)\cdot\frac{m^2}{\cancel{sec^2}}}{\cancel{\frac{1}{sec^2}}} = \frac{1}{m} \checkmark$$

CONCLUSION:

① as frequency increases (ν ↑) β decreases

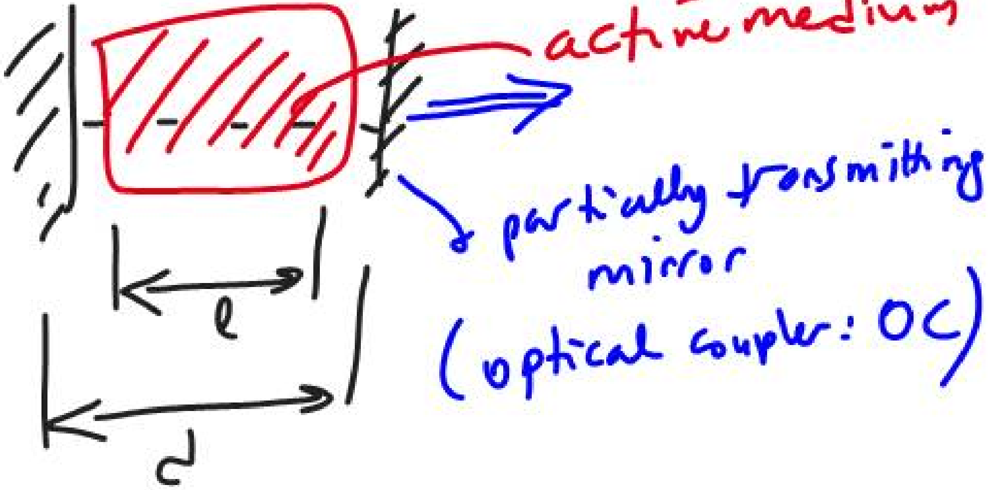
The higher the ν the lower the gain (β)

It is more difficult to obtain lasers in higher frequencies than to obtain lasers in lower frequencies

⇒ Easier to LASE in IR region than to LASE in UV region.

② $N_2 > N_1$ to have a $\beta > 0$ Population inversion

Power output for CW operation



P_0 initial power

$$P_0 + P_0 \cdot e^{(\beta-\alpha)2l}$$

$$P_0\left[1+e^{(\beta-\alpha)2l}\right]$$ after one round-trip

The power output of the laser will be:

$$P = T \cdot P_0\left[1+e^{(\beta-\alpha)2l}\right]$$

$T = 1 - R$: transmittance of the output mirror (OC) another round-trip:

$$P = T P_0 \left[1 + e^{(\beta-\alpha)2l} + e^{2(\beta-\alpha)2l} \right]$$

$$P = T P_0 \sum_{n=0}^{\infty} e^{n(\beta-\alpha)2l}$$

if $n(\beta-\alpha)2l < 0$ for stable operation (i.e. power should not increase indefinitely)

$$\boxed{P = \frac{T P_0}{1 - e^{(\beta-\alpha)2l}}}$$

- When $\beta = \alpha$ $P \to \infty$
- Therefore for continuous wave (cw) stable operation β should be slightly α less than α.

Since β shows the gain for stimulated emission.
If β is slightly less than α, the missing part
of the gain is substituted by spontaneous emission
which is always there with stimulated emission

By this way, continuous wave (cw)
operation is possible.